Iterated Total Time on Test Transforms Comparison: k-mart stochastic modelling

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Abstract

Iterated Total Time on Test (TTT) transforms can be used to characterize certain stochastic orderings. In particular, orderings with respect to expectation of functions, convex with respect to polynomials, are equivalent to higher order TTT domination under certain conditions. This ordering gives rise to a k-mart structure, which gives meaningful representation for the relationship of two jointly distributed random variables. Here we apply these ideas to the construction of stochastic models.

1. Introduction

Often times, the observed variable Y is modeled in terms of an unobserved random variable X and an error term ε. Some desired features of a stochastic model $Y = X + \varepsilon$ are:

1. $E(\varepsilon) = 0$, and hence $E(X) = E(Y)$
2. $\text{cov}(X, \varepsilon) = 0$

A model with this characteristic has a martingale structure, i.e. (X,Y) are jointly distributed random variables with $E(Y|X) = X$. To see this, let $\varepsilon = Y - X$. Then

$E(\varepsilon) = E(Y) - E(X) = E(E(Y|X)) - E(X) = 0$ and
$\text{cov}(X, \varepsilon) = \text{cov}(X, Y) - \text{cov}(X, X) = E(XY) - E(X)E(Y) - \text{var}(X) = E(X^2) - [E(X)]^2 - \text{var}(X) = 0$

Note that (X,Y) has a martingale structure if and only if the expectation of any convex function evaluated at Y is greater than or equal to the expectation of the same function evaluated at X. In this case, Y is said to be a *dilation* of X (Blackwell, 1951 and 1953, Strassen, 1965).

What diagnostics tool can be used to detect dilations? If a survival function $\overline{F}$ with finite moment is integrated from t to ∞ and scaled by its mean, a new survival function is produced, $\overline{T}_F$. The cumulative distribution function corresponding to this new survival function $T_F = 1 - \overline{T}_F$ is known as the *Total Time on Test* (TTT) transform (Barlow, Campo, 1975).

Two distributions F and G with the same mean are ordered such that the TTT of one dominates the other ($\overline{T}_G \geq \overline{T}_F$) if and only if there exists random variables X and Y with distribution F and Y respectively, such that Y is a dilation of X (Ross, 1983, Section 8.5, Shaked and Shanthikumar, 1994, Section 2.A).

2. Generalized convexity, Iterated TTT, k-mart

Since the TTT defines a new distribution, it can be applied iteratively. What happens when the iterated transform of a distribution dominates the other? Is there a corresponding notion for dilation? Is there a generalization of the martingale structure?

2.1 Generalized convexity

The classical concept of convexity can be viewed as the comparison of a real function with a straight line. A function f is convex if and only if for every line l that intercepts the function in exactly two points, f - l has two sign changes with the last sign change being -, +. An equivalent approach for differentiable functions is that it is convex if and only if the second derivative is nonnegative (where it exists).

A function can be convex with respect to a polynomial P (instead of a line) of degree m. In this case, the number of sign changes of f - P is m + 1, and the last sign change is -, +. It can be shown that the (m + 1)th-derivative of f being nonnegative is equivalent to being convex with respect to a polynomial of degree m (Roberts and Varberg, 1973). We will refer to these functions as *m-convex*.

2.2 Iterated TTT

As mentioned before, the TTT transform of the distribution F defines a new distribution denoted by T_F, provided that the first moment exists. This new distribution is referred to as the *first order* TTT transform. The transform is then applied again to the first order TTT transform to get the *second order* TTT transform, which is a distribution if the second moment of the original distribution is finite. This process can be extended iteratively. We will denote the m-order transform of F as T_F^m, which is a valid distribution if the first m moments of F are finite

2.3 K-mart

What is the generalization of a martingale structure? A *k-mart*. The jointly distributed random variables $\left(X_1,\ldots,X_k,Y\right)$, with $\left(X_1,\ldots,X_k\right)$ iid, are said to have a k-mart structure if and only if

$$E\left(Y^j \mid X_1,\ldots,X_k\right)=\frac{X_1^j,\ldots,X_k^j}{k}$$

for $j=1,\ldots,m$, where m is a positive integer such $k=\left[\dfrac{m+2}{2}\right]$, $[\cdot]$ denotes the greatest integer function.

2.4 A relationship

Domination of the m-order TTT transform characterizes a *balayage* which is a stochastic ordering of two distributions defined by expectations of m-convex functions (Meyer, 1966).

Theorem 2.4.1. Let F and G be the distribution functions of the random variables X and Y respectively. G is a balayage of F, i.e. $E\left(c\left(Y\right)\right)\geq E\left(c\left(X\right)\right)$ for every m-convex function c, if and only if the first m-moments of F and G are equal and $\overline{T}_G^m \geq \overline{T}_F^m$ (Vera and Lynch, 2004). In symbols, $G\overset{m}{>}F$ or $Y\overset{m}{>}X$.

Theorem 2.4.2. Let F and G be distribution functions. G is a balayage of F, i.e. $G\overset{m}{>}F$ if and only if there exists jointly distributed random variables $\left(X_1,\ldots,X_k,Y\right)$ with $X_i \sim F$, $Y \sim G$, $k=\left[\dfrac{m+2}{2}\right]$, having a k-mart structure.

2.5 Diagnostics

A way to discern a k-mart structure between two distributions F and G, is to look at the number of sign changes of F - G, denoted by s. The j-th iteration of the TTT transform has at most s - j sign changes, provided the first j moments of F and G are the same. This fact can be used to recognize a balayage by the use of iterated TTT transforms. In particular, $G \overset{m}{>} F$ if G - F has m sign changes.

3. Illustration: Mixtures of exponential distributions.

The mixture of an exponential distribution is a balayage of another mixed exponential, thus imposing a k-mart structure.

Let $h(\cdot;\beta)$ denote the density of an exponential distribution with mean β, and let $H(\cdot;\beta)$ be the corresponding cdf. Let B be a mixing distribution over the parameter β, and let C be a distribution with $B \overset{m}{>} C$. Let $G = \int H(\cdot;\beta)\, dB(\beta)$ and $F = \int H(\cdot;\beta)\, dC(\beta)$. Under this conditions, $G \overset{m}{>} F$ and thus a k-mart structure can be imposed between the two mixtures.

To see this, notice that since the first m moments of B and C are equal,

$$\int x^j dF(x) = \int x^j \int h(x;\beta)\, dB(\beta) dx = \int\int x^j h(x;\beta)\, dx dB(\beta) = j! \int \beta^j dB(\beta)$$

$$j! \int \beta^j dC(\beta) = \int\int x^j h(x;\beta)\, dx dC(\beta) = \int x^j \int h(x;\beta)\, dB(\beta) dx = \int x^j dF(x).$$

Thus the first m-moments of F and G are the same.

Further, let us assume that dB - dC has m + 1 sign changes. Then

$$(g - f)(x) = \int h(x;\beta)\big(dB(\beta) - dC(\beta)\big) = \int \frac{1}{\beta} e^{-x/\beta}\big(dB(\beta) - dC(\beta)\big)$$

has at most m + 1 sign changes, since the exponential distribution has a kernel that is totally positive of all orders in x and β (see the Variation Diminishing Theorem in Karlin, 1968). The fact that the two densities integrate to one, implies that G - F has at most k sign changes (the integral of the difference is zero). Hence, there is domination of the m-order TTT transform of G over F, i.e. $\overline{T}_G^m \geq \overline{T}_F^m$. Figure 1 compares a 4 point mixture with a 2 point mixture of exponential distributions.

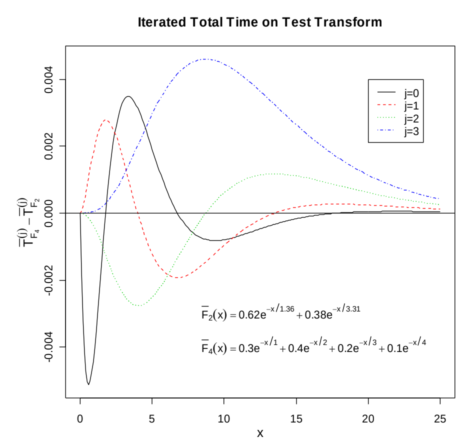

Figure 1: Comparison of higher order TTT transforms for mixed exponential distributions

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